The CRISP Consortium

Continuum Mechanics

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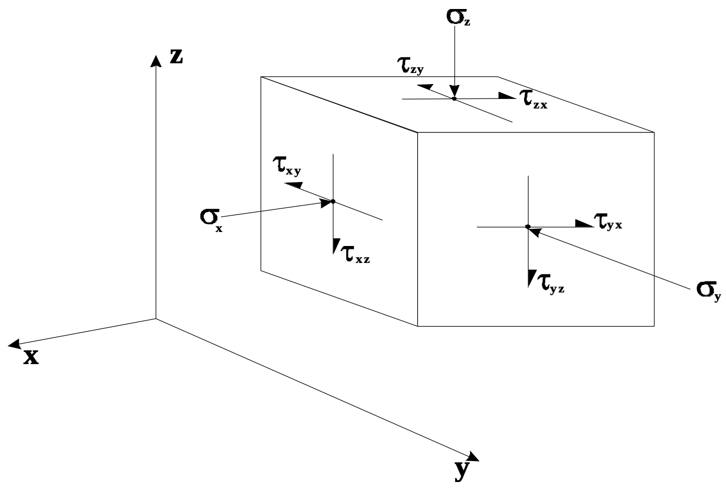
Produced by the CRISP Consortium Ltd

THEORY OF STRESSES

In a three dimensional loaded body, there are six independent components of stress at a point

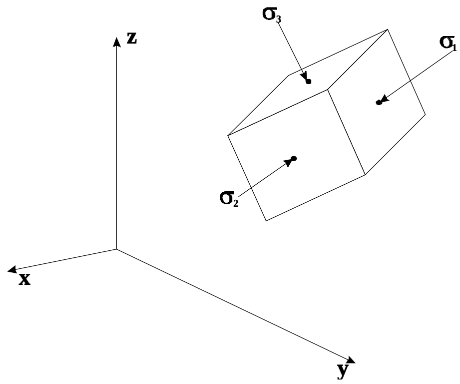


Shear stress notation: τ_{ab}

face of cube (a), direction (b)

By moment equilibrium:

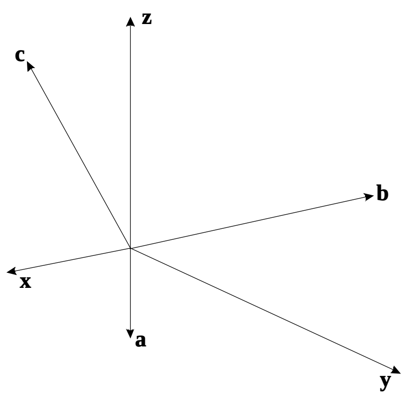
$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx}$$



In any stressed material there are always three mutually perpendicular planes on which the shear stresses are zero.

These are called the <u>principal</u> planes.

The direct stresses acting on these planes are called <u>principal stresses</u> (σ_1, σ_2, σ_3).



In a stressed body the components of stress are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx}.

We can also describe the stress state with respect to another set of axes (e.g. a, b, c).

Stress invariants are functions of the stress components which are independent of the axis system chose. For example,

$$p = \frac{s_x + s_y + s_z}{3} = \frac{s_a + s_b + s_c}{3}$$

is a stress invariant.

Another stress invariant is q where:

$$q = \sqrt{\frac{(s_x - s_y)^2 + (s_y - s_z)^2 + (s_z - s_x)^2 + 6t_{xy}^{\ 2} + 6t_{yz}^{\ 2} + 6t_{zx}^2}{2}}$$

$$= \sqrt{\frac{(s_a - s_b)^2 + (s_b - s_c)^2 + (s_c - s_a)^2 + 6t_{ab}^{\ 2} + 6t_{bc}^{\ 2} + 6t_{ca}^{\ 2}}{2}}$$

Note that these general definitions of p and q reduce to those given later for triaxial stress conditions (see lecture on Cam-clay).

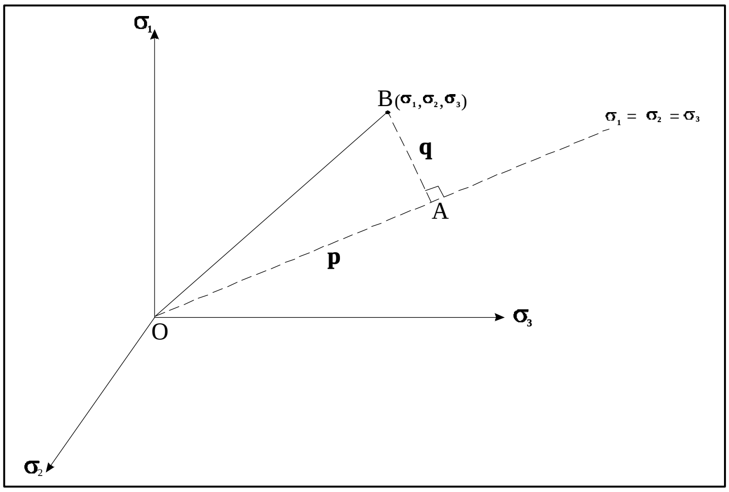
We will encounter these invariants more frequently expressed in terms of the principal stresses, i.e.

$$p = \frac{s_1 + s_2 + s_3}{3}$$

$$q = \sqrt{\frac{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2}{2}}$$

Note σ_1, σ_2, σ_3 are also stress invariants.

There is a simple geometric interpretation of p and q in principal stress space.



The line $\sigma_1 = \sigma_2 = \sigma_3$ is called the “space diagonal” or the “hydrostatic axis”

$$OA = \frac{s_1 + s_2 + s_3}{\sqrt{3}} = \sqrt{3}\,\mathrm{p}$$

$$AB = \sqrt{\frac{2}{3}\left(s_1^{\ 2} + s_2^{\ 2} + s_3^{\ 2} - s_1 s_2 - s_2 s_3 - s_3 s_1\right)}$$

$$AB = \sqrt{\frac{2}{3}}\,\mathrm{q}$$

ELASTICITY

Generalised Hooke's Law:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx}$$

PLASTICITY

INTRODUCTION

The theory of elasticity allows the calculation of stresses and strains in a loaded body when the body is linear and elastic.

The theory of plasticity allows the calculation of stresses and strains in a loaded body when plastic yielding takes place.

History (Elasticity)

1822 Cauchy invents concepts of stress and strain

1885 Boussinesq gives his solutions for elastic stress distributions (still used in geotechnical engineering today)

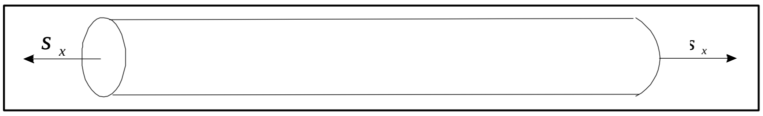
(For 100 years mathematicians slave away obtaining elasticity solutions...)

1956 Finite element method invented (mathematicians are now redundant)

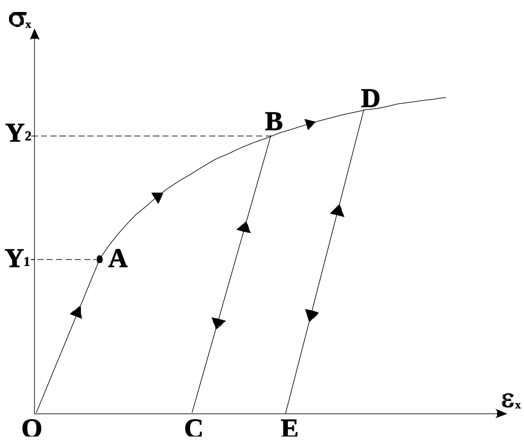
<u>History</u> (Plasticity)

1773	Coulomb identifies two components in the strength of soil - cohesion and friction
1864 (Tresca) 1913 (Von Mises)	Criteria are put forward for limits to the elastic behaviour of <u>metals</u>
1940's	3 types of statement are now seen to be necessary to completely describe plastic stress-strain relations: a) yield criteria b) flow rule c) hardening law
1950	R. Hill's book "Mathematical Theory of Plasticity" is published.

PLASTIC BEHAVIOUR FOR ONE- DIMENSIONAL LOADING



Direct strain in x direction is ε_x.



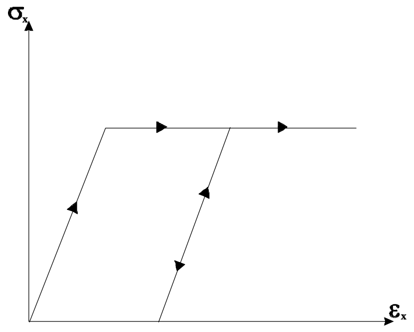
Typical stress -strain relation for an elastic, work hardening plastic material (e.g. metal alloy)

OA is elastic. A is a yield point. Y_1 is the uniaxial yield stress.

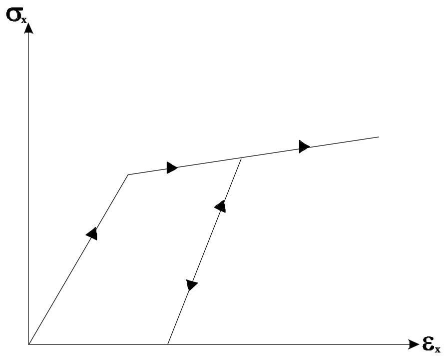
BC and DE are elastic unloading and reloading (parallel to OA).

On reloading to B the yield stress has increased to Y_2. The material is harder and has "strain (or work) hardened".

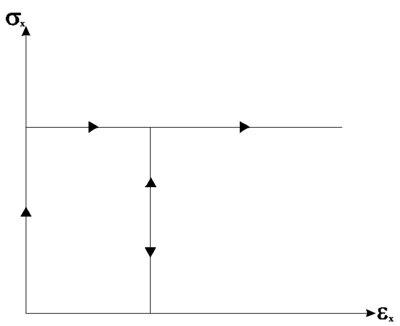
In describing plastic behaviour, the following simplifications ("idealisations") are often made.



Elastic, perfectly plastic
(steel behaviour is quite like this)

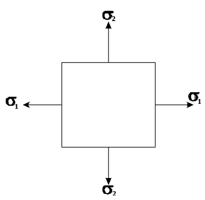


Elastic, linear strain hardening plastic



Rigid-plastic
(often assumed in collapse calculations)

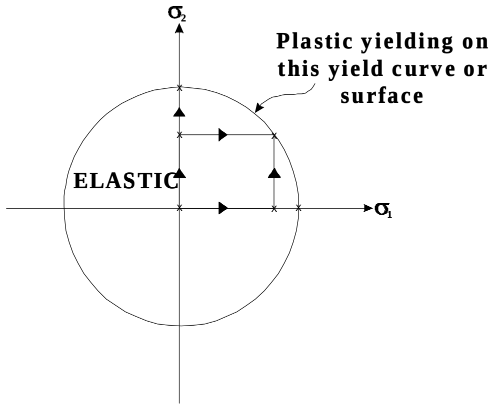
YIELD FUNCTIONS - BASIC IDEAS



σ_2 is held constant and σ_1 is increased until yielding starts (or vice versa)

The combinations of σ_1 and σ_2 that cause yielding are described by a yield function.

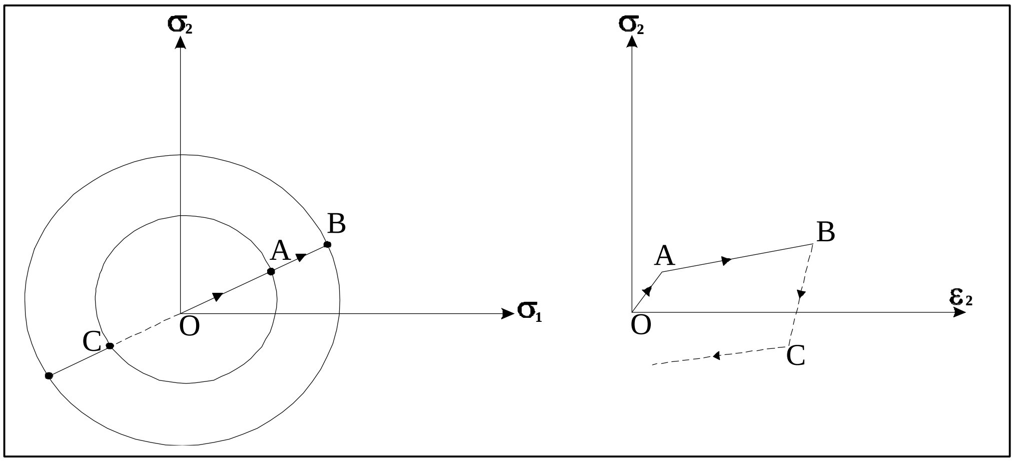
Yield functions can be represented by lines in 2D stress space and surfaces in 3D stress space.



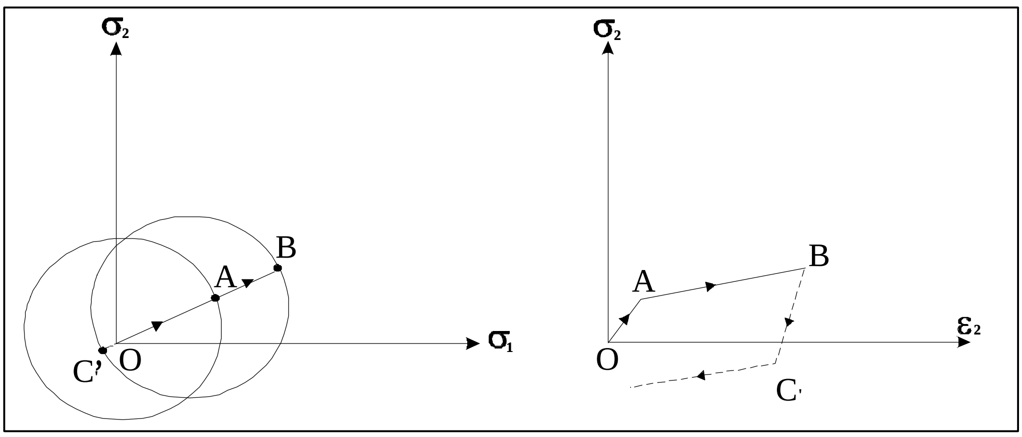
For elastic-perfectly plastic behaviour:

Stress states inside the yield surface are elastic.
Stress states outside the yield surface are impossible, by definition.

If strain hardening takes place, there are two possibilities:



The yield surface expands uniformly - this is called <u>isotropic hardening</u>



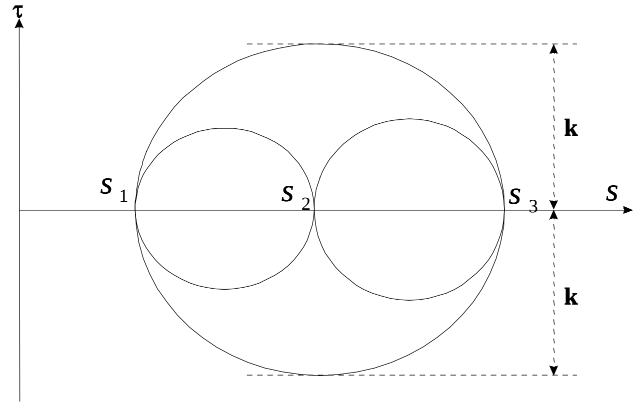
The yield surface is “dragged along” - this is called <u>kinematic hardening</u>.
The differences between these two assumptions become important if unloading takes place.

YIELD FUNCTIONS - EXAMPLES

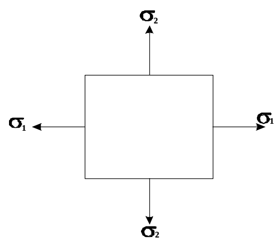
a) TRESCA

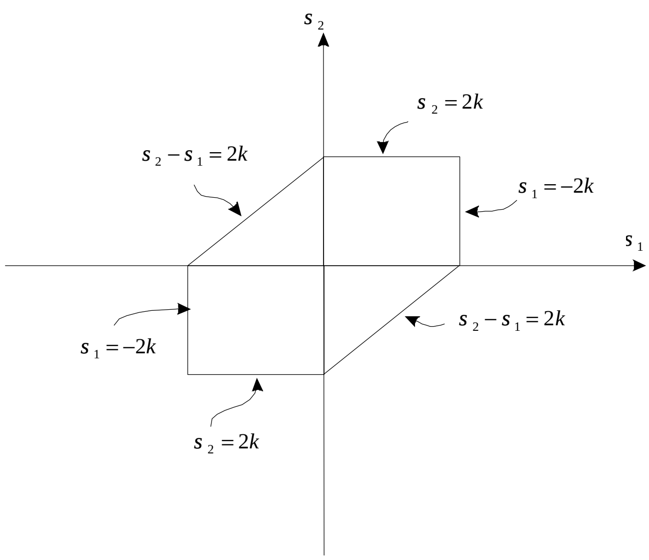
Yielding takes place when the maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2-\sigma_3|$, $|\sigma_3-\sigma_1|$ equals a critical value (2k).

This can be interpreted (by considering Mohr's circles) as being equivalent to limiting the maximum shear stress on any plane in the material to being less than or equal to k.

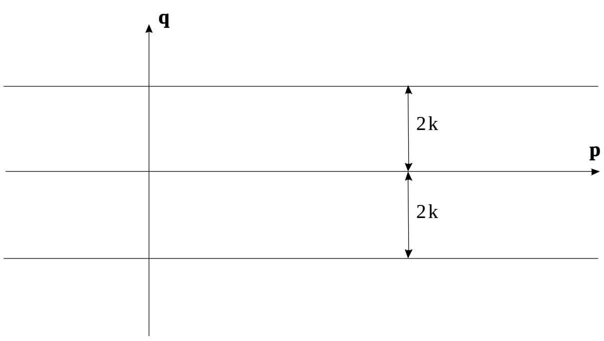


Consider the yield condition in (σ_1, σ_2) stress space when σ_3 = 0.





For triaxial stress conditions σ_2 = σ_3 and we can plot the yield function in (p, q) space.



b) VON MISES

Plastic yielding takes place when:

$$(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2 = 2Y^2$$

where Y is the yield stress in uniaxial tension.

Note (from our earlier definition of q) that the above equation can be written as:

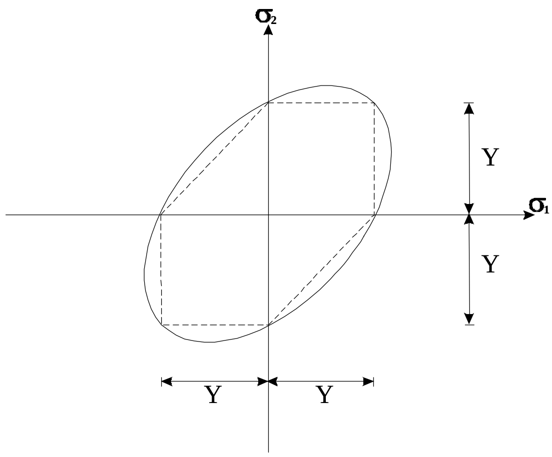
$$q = Y$$

Structural and mechanical engineers call the stress parameter q either:

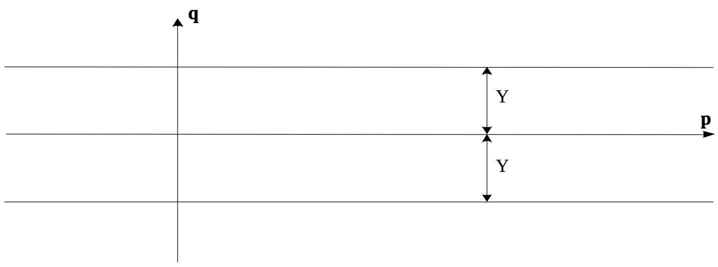
i) the Von Mises stress,
ii) the Von Mises equivalent stress, or
iii) the equivalent stress

and use the symbol σ_E rather than q.

Von Mises yield function in (σ_1, σ_2) space when $\sigma_3 = 0$:

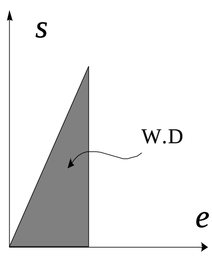


This is an ellipse with its major axis along $\sigma_1 = \sigma_2$. The Tresca yield criterion is shown thus ----- for comparison. Clearly Y = 2k.
In triaxial stress conditions $\sigma_2 = \sigma_3$ and the yield function in (p, q) space is as shown:



STRAINS

For every stress component or invariant there is a "corresponding" strain component so that the work done per unit volume in elastic deformation is =1/2 stress x strain.



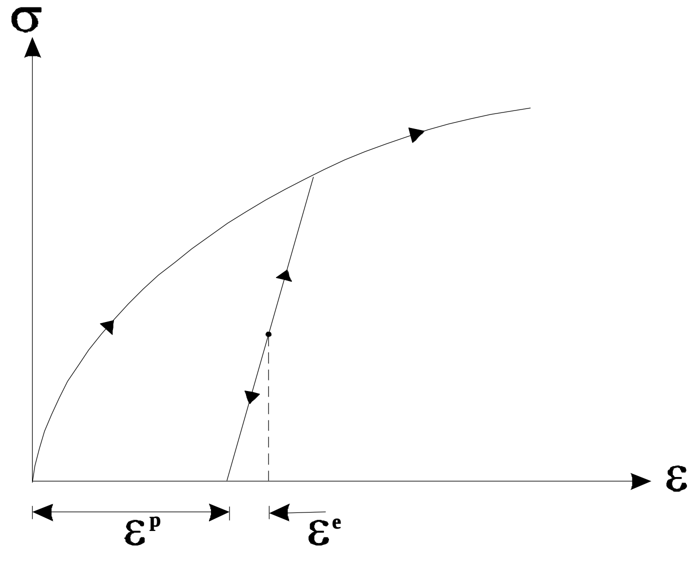
Stress	**Strain**
σ_x	ε_x
σ_y	ε_y
σ_z	ε_z
τ_{xy}	γ_{xy}
τ_{yz}	γ_{yz}
τ_{zx}	γ_{zx}
σ_1	ε_1
σ_2	ε_2
σ_3	ε_3

Stress invariant	Strain invariant
p (defined previously)	ε_p (volumetric strain)
$=\dfrac{s_x + s_y + s_z}{3}$	$= e_x + e_y + e_z$
$=\dfrac{s_1 + s_2 + s_3}{3}$	$= e_1 + e_2 + e_3$
q (defined previously)	ε_q (deviatoric strain)

$$e_q = \frac{2}{3}\sqrt{\frac{(e_x - e_y)^2 + (e_y - e_z)^2 + (e_z - e_x)^2 + \frac{3}{2}\left(g_{xy}{}^2 + g_{yz}{}^2 + g_{zx}{}^2\right)}{2}}$$

$$e_q = \frac{2}{3}\sqrt{\frac{(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2}{2}}$$

PLASTIC STRAINS



The plastic strain is the strain which remains on completely unloading the applied stress.

$$\varepsilon^p + \varepsilon^e = \varepsilon^T$$

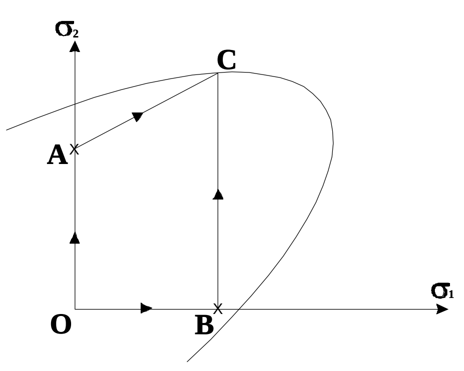
ε^p = plastic strain
ε^e = elastic strain
ε^T = total strain

CALCULATION OF ELASTIC STRAINS

Suppose that the soil is in some stress state $\underline{\sigma}$ (= $[\sigma_1, \sigma_2, \sigma_3]^T$). An increment of stress is now applied to the soil ($\Delta\underline{\sigma}$). The resulting strains, if the behaviour is elastic, can be calculated as $\Delta\underline{\varepsilon} = \mathbf{C}\Delta\underline{\sigma}$ where **C** is a square matrix containing elastic constants.

CALCULATION OF PLASTIC STRAINS

In contrast to elastic behaviour, it is found that plastic strains are strongly dependent on the current total stresses.



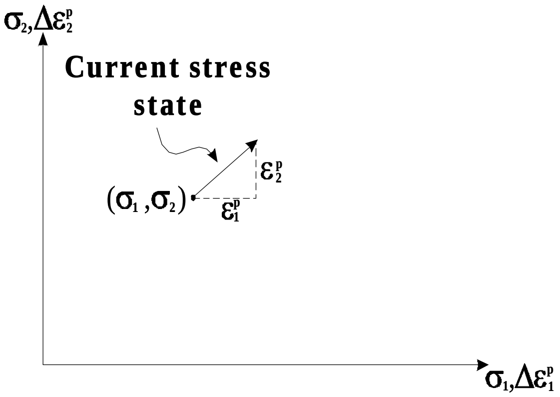
Thus the plastic strains generated a yield point C are not dependent on the direction of the stress path when yielding starts.

i.e. paths OAC and OBC lead to the same plastic strains.

PLASTIC POTENTIALS

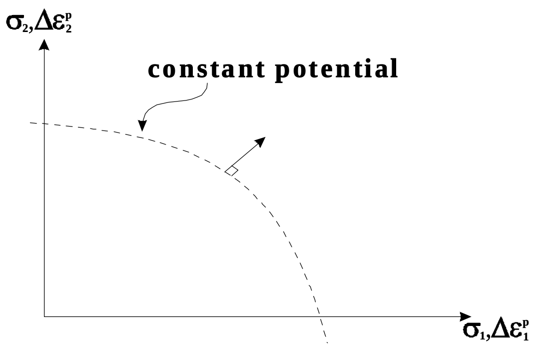
Mathematically, the type of behaviour described above can conveniently be expressed in terms of a potential function such that the derivatives of the potential function define the ratios of the plastic strains.

Plastic strain components are often plotted in stress space:



often called the strain increment vector

Mathematically we write:



$$d\underline{e}^p = m\frac{\partial G}{\partial \underline{s}} \qquad (1)$$

where G is the plastic potential

For metals the plastic potential function is the same as the yield function i.e. G ≡ F and hence we can write:

$$d\underline{e}^p = m\frac{\partial F}{\partial \underline{s}} \qquad (2)$$

(m is a scalar number which depends on the hardening law and the details of a particular analysis. It is called the plastic multiplier).

when G ≡ F we say there is normality (the strain increment vector is normal to the yield surface) - also called associated flow.

Equations like (1) and (2) above are called flow rules.

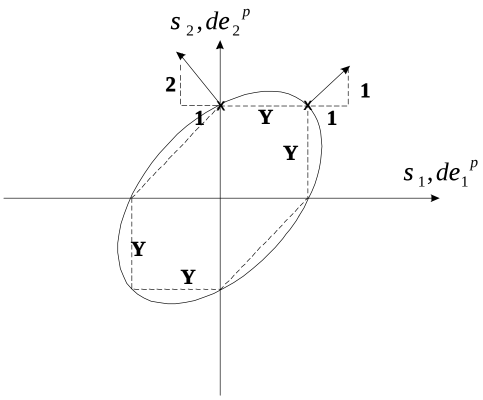
They govern the ratios of the plastic strain components.

Flow Rule - Example with Von Mises

(a) in σ_1, σ_2 space when $\sigma_3 = 0$

$$F(\sigma_1, \sigma_2) = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 - Y^2 = 0$$

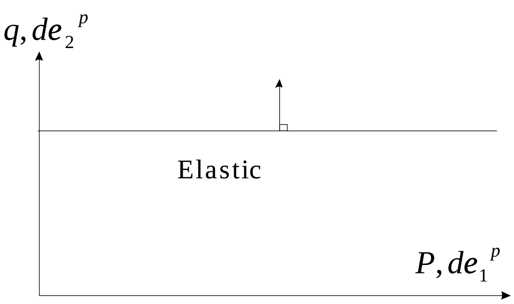
$$\begin{bmatrix} \delta\varepsilon_1^{\,p} \\ \delta\varepsilon_2^{\,p} \end{bmatrix} = \begin{bmatrix} \mu\dfrac{\partial F}{\partial \sigma_1} \\ \mu\dfrac{\partial F}{\partial \sigma_2} \end{bmatrix} = \begin{bmatrix} \mu(2\sigma_1 - \sigma_2) \\ \mu(2\sigma_2 - \sigma_1) \end{bmatrix}$$



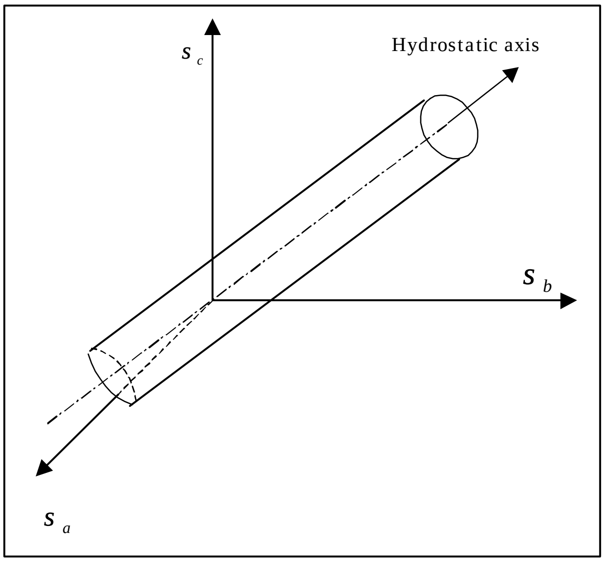
By inspection the strain increment vector is perpendicular to the ellipse previously described.

(b) in p, q space

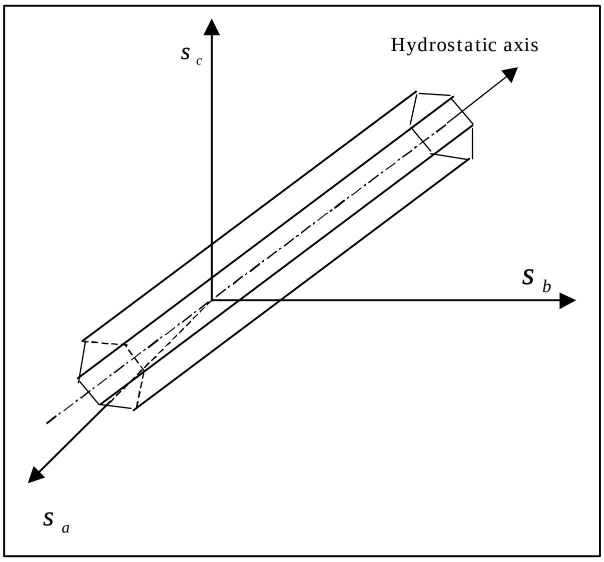
Yielding takes place with zero volumetric strains $\left(\delta\varepsilon_p^{\,p} = 0\right)$



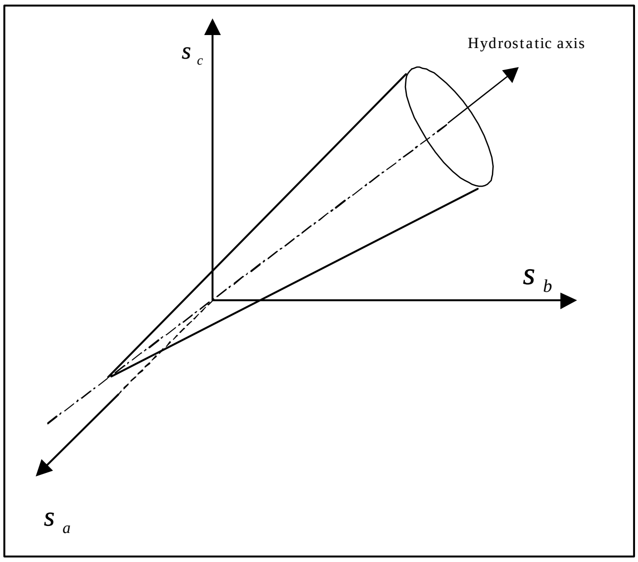
(1) <u>The Von Mises Yield Surface</u>



(2) <u>The Tresca Yield Surface</u>



(3) <u>The Drucker-Prager Yield Surface</u>



(4) <u>The Mohr-Coulomb Yield Surface</u>