B-Bar method, Selective integration technique for near incompressible materials

Introduction

In the analysis of incompressible materials (materials with poisson's ratio =0.49, undrained problems, or plastic zones), an overstiff response has been observed in many cases. Nagtegaal, et, al, [1] proposed a criterion for determining the suitability of finite elements for incompressible conditions. They evaluate the ratio of the total number of degrees of freedom to the total number of constraints at the integration points. If this ratio is greater than 1, the finite element is suitable for material incompressibility analysis. If the ratio is less than 1, the finite element is deemed unsuitable and may produce "locking" effect.
Sloan and Randolph [2] demonstrated that as the order of the polynomial defining the displacements within an element is increased, the increase of new degrees of freedom is greater than the increase in the incompressibility constraints. Therefore, higher order elements are less likely to produce an overstiff response.

Solutions for incompressible elasticity

The ideal solution for elements which suffer from incompressibility constraints is to break up the stiffness matrix into two parts, one dilatational and one deviatoric. It is the dilatational part which produces the overstiff response. This part can then be integrated using a lower order integration rule. This method is called **selective integration method**.

The element stiffness is given as

$$K^e = \int_{\Omega^e} B^T DB \, d\Omega$$

The element internal force vector is given as

$$f^e = \int_{\Omega^e} B^T \sigma \, d\Omega$$

$$B = \lfloor B_1, \quad B_2, \quad \ldots \quad , \quad B_{n_{en}} \rfloor$$

where n_{en} is the number of element nodes

$$B_a = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \\ 0 & B_3 & B_2 \\ B_3 & 0 & B_1 \\ B_2 & B_1 & 0 \end{bmatrix}$$

in which $B_i = \dfrac{\partial N_a}{\partial x_i} \qquad 1 \le i \le 3$

where N_a is the shape function associated with node a and x_i is the i^{th} Cartesian coordinate.

With the selective integration method, the stress tensor is split into deviatoric part and dilatational (mean) part, so that the stiffness equation becomes:

$$\sigma^{dev} = K^{dev} \varepsilon^{dev}$$
$$\sigma^{dil} = K^{dil} \varepsilon^{dil}$$

$$D^{dil} = K_{bulk} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D^{dev} = 2G \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K^e = K^{dev} + K^{dil}$$

$$K^{dev} = \int_{\Omega^e} B^T D^{dev} B \, d\Omega$$

$$K^{dil} = \int_{\Omega^e} B^T D^{dil} B \, d\Omega$$

$$K^{dev} = \sum_{ip}^{ngp} B^T D^{dev} B \, w_i \, dv$$

where ngp is the number of integration points for full integration

$$K^{dil} = \sum_{ip}^{rgp} B^T D^{dil} B \, w_i \, dv$$

where rgp is the number of integration points for reduced integration. For example, an eight nodded quadrilateral would have 3x3 integration points for full integration and 2x2 points for reduced integration.

B-Bar method for general incompressibility problems

The method of selective integrations works well for isotropic elastic materials in which it is easy to split up the stress into deviatoric and dilatational parts. However, with elasto-plastic materials this procedure is not so straightforward. An alternative method is to split the B matrix into dilatational and deviatoric parts. This method is also referred as the B-Bar method. See Hughes T.J.R. [3].

Let B_a^{dil} denote the dilational (mean) part of B_a, ie

$$B_a = \frac{1}{3}\begin{bmatrix} B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deviatoric part of Ba is then defined by

$$B_a^{dev} = B_a - B_a^{dil}$$

$$\overline{B}_a^{dil} = \frac{1}{3}\begin{bmatrix} \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B}_a = B_a^{dev} + \overline{B}_a^{dil}$$

In details, this is given as:

$$\overline{B}_a = \begin{bmatrix} B_5 & B_6 & B_8 \\ B_4 & B_7 & B_8 \\ B_4 & B_6 & B_9 \\ 0 & B_3 & B_2 \\ B_3 & 0 & B_1 \\ B_2 & B_1 & 0 \end{bmatrix}$$

where

$$B_4 = \frac{\overline{B}_1 - B_1}{3} \qquad B_5 = B_1 + B_4$$

$$B_6 = \frac{\overline{B}_2 - B_2}{3} \qquad B_7 = B_2 + B_6$$

$$B_8 = \frac{\overline{B}_3 - B_3}{3} \qquad B_9 = B_3 + B_8$$

The whole approach reduced to appropriate definitions of the $\overline{B}_i's$. Various examples are presented by Hughes [3]. The following definition is used in this research:

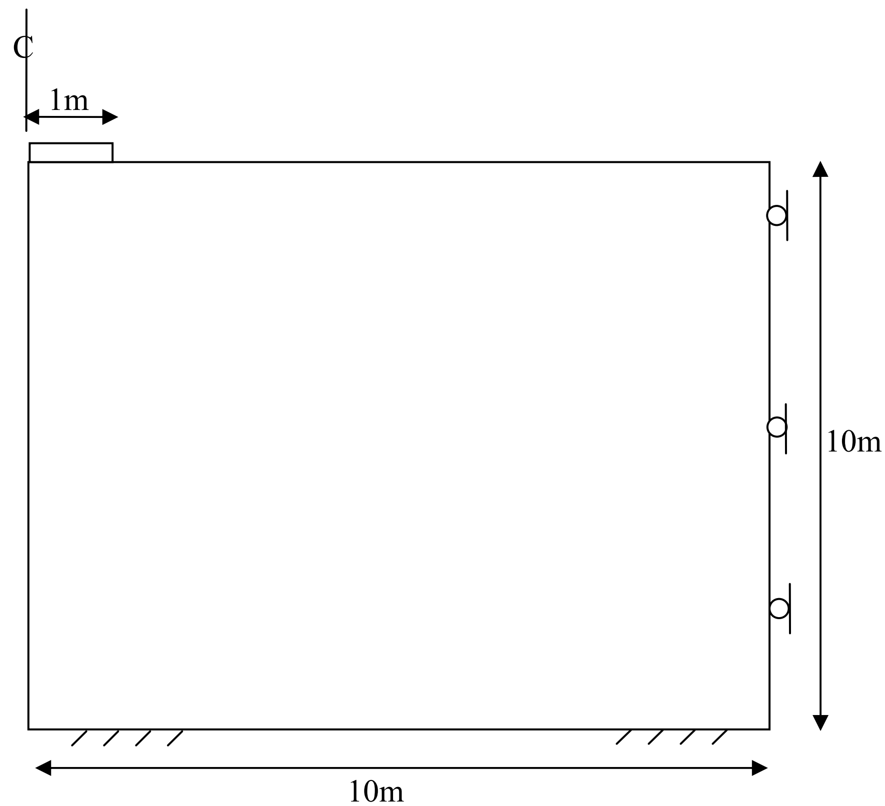
$$\overline{B}_i = \frac{\int_{\Omega^e} B_i d\Omega}{\int_{\Omega^e} d\Omega}$$

The denominator represents the finite element's volume

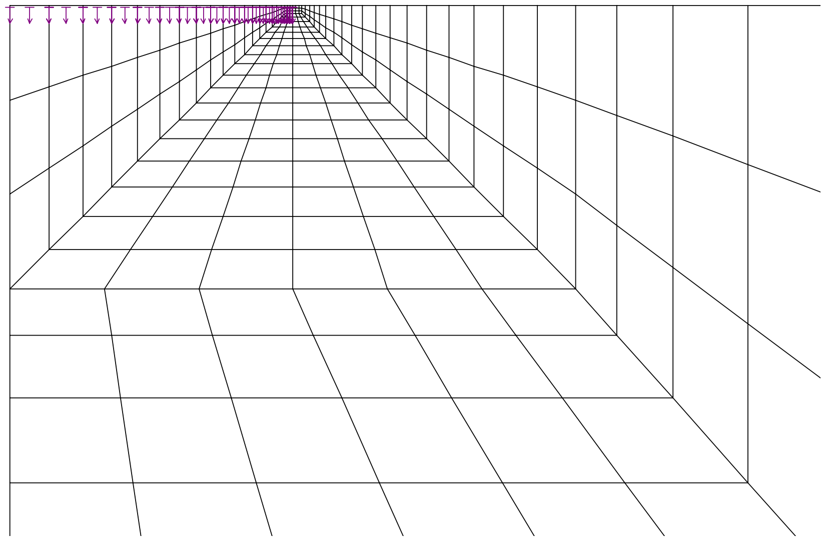
The above equations present the B matrix for 3D case. For the 2D plane strain case, the B matrix is modified by eliminating the in-plane strain and the two extra shear strains. For the special case of Axi-symmetric, the hoop strain is another direct strain. However, there is only one shear strain in the axi-symmetric case.

Example of Axi-symmetric analysis of a smooth footing on Tresca (cohesive) soil)

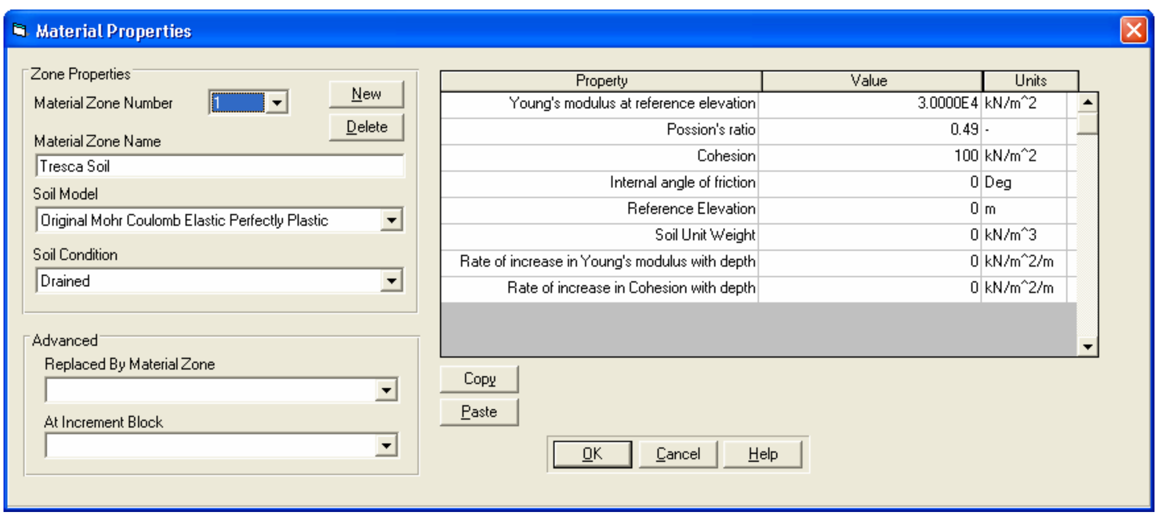
Mesh outline:



Mesh details near footing area:
The mesh was generated using a two-way structured mesh



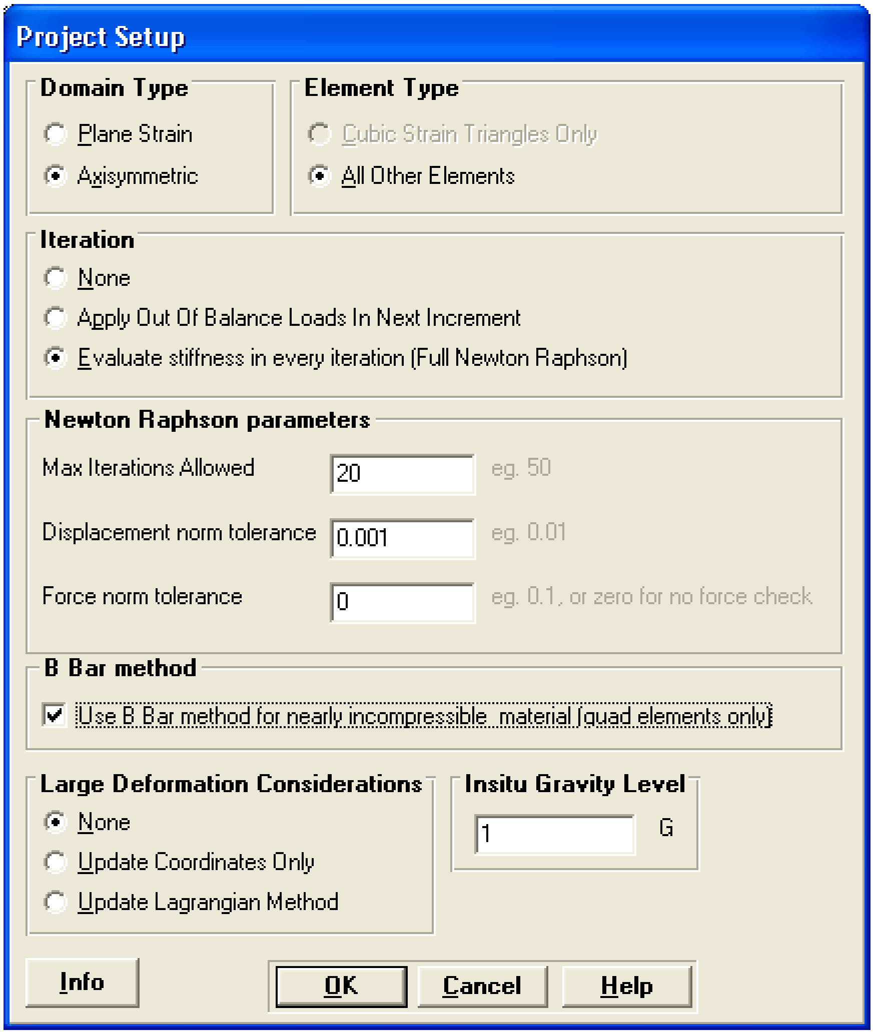
Material properties:



Bearing capacity versus normalized displacement for the normal case (8 noded quads with full integration), and the BBar method.

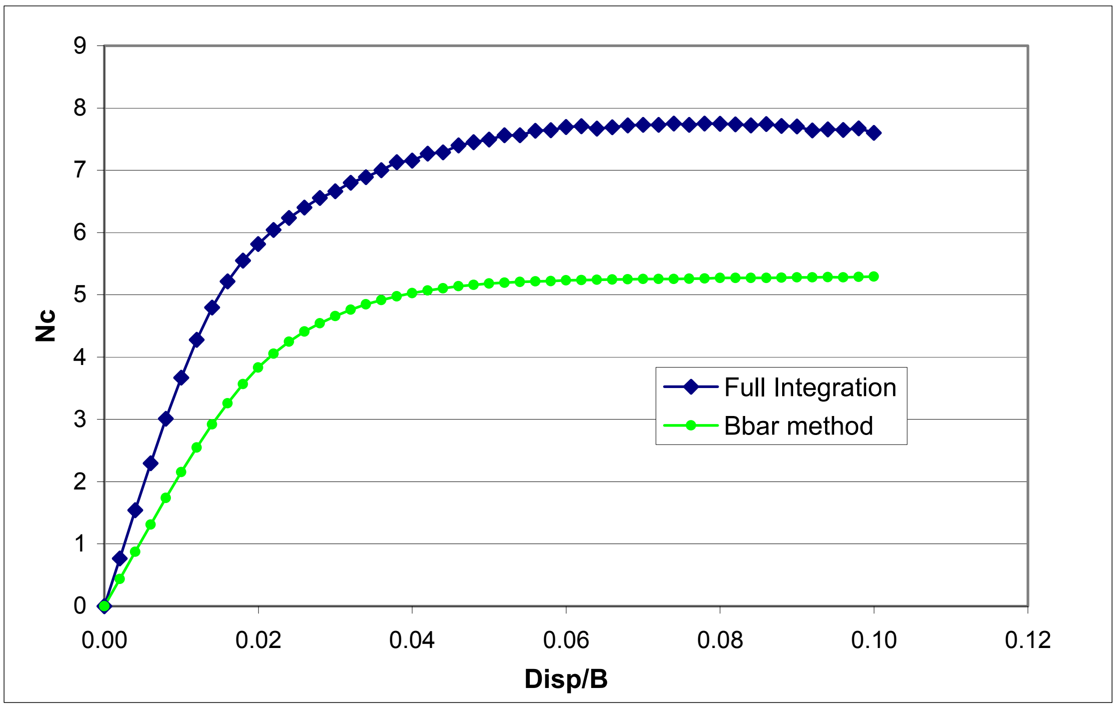
Setting up the BBar option.

This is a new feature in Project Setup menu of SAGE-CRISP version 5.2. Open the Project Setup Menu and check the box for BBar method as shown below.



Plotting the results

The reactions for a set of applied displacements are summed up and tabulated in a file identified by the analysis ID and with the ending **_rec.txt.** Open this file with MS Excel and plot the graph of applied displacement against reaction as shown below. In the graph below, the total reaction is divided by the circular area of the footing and normilised by the cohesion of the material. The displacement is also normilised by the radius of the circular footing.



References

1- Nagtegaal, J.C, Park, D.M., Rice, J.R., "On numerically accurate finite element solutions in the fully plastic range" Comput. Methods Appl. Mech. Eng. **4** 153-177 (1974).

2- Sloan, S.W, Randolph, M. F, "Numerical prediction of collapse loads using finite element method" Int. J. Numer. Anal. Methods Geomech, **6** 47-76 1982.

3- Thomas Hughes J.R., "The Finite Element Method" Dover Publications, Inc. New York, 2000.